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A Spinor from Semiderivatives

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Abstract Fractional derivatives have been known since the time of Leibniz and have been used in various branches of physics. The present paper shows how they can be used to generate a spinor field, much as the gradient operator generates a vector field. These spinor fields are zero kinetic energy solutions to the Dirac equation.

Keywords Half derivatives · Spinors · Dirac equation

1 Introduction

Fractional derivatives, introduced by Leibniz, are derivatives whose order is a noninteger. They have many applications; one of the most common is in the theory of diffusion. An excellent overview of the subject with bibliography is given in Oldham and Spanier [1]. More recent references can be found in Bayin [2]. In this paper, I show how fractional derivatives can be used to construct a spinor field which is a solution of the Dirac equation.

2 The Spinor Field

Let S be a unitary matrix given by:

$$S = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}i/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}i/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let coordinates u and v be defined by $[u, v, z]^T = S[x, y, z]$, so that

$$u = \sqrt{2}/2(x + iy) \tag{1}$$

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and

$$v = \sqrt{2}/2(-x + iy). \tag{2}$$

We can show that if $f(x, y, z)$ is a solution to Laplace's equation, then the quantity ψ defined by $\psi = (\partial^{1/2} f/\partial^{1/2} u, -\partial^{1/2} f/\partial^{1/2} v)$ is a spinor. We must show that the matrices which transform ψ under rotations about the x, y and z axes are given by $R_x^{(1/2)} = \exp(i\alpha\sigma_x)$, $R_y^{(1/2)} = \exp(i\beta\sigma_y)$, and $R_z^{(1/2)} = \exp(i\gamma\sigma_z)$, where α, β, and γ are the respective angles of rotation and σ_x, σ_y and σ_z are the Pauli spin matrices. The superscript of (1/2) signifies a spinor transformation. When the Pauli matrices, given by: $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, are plugged in to the formulas for the $R^{(1/2)}$ matrices, we obtain [3]:

$$R_x^{(1/2)}(\alpha) = \exp(i\sigma_x\alpha/2) = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}, \tag{3}$$

$$R_y^{(1/2)}(\beta) = \exp(i\sigma_y\beta/2) = \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix}, \tag{4}$$

$$R_z^{(1/2)}(\gamma) = \exp(i\sigma_x\gamma/2) = \begin{pmatrix} \exp(i\gamma/2) & 0 \\ 0 & \exp(-i\gamma/2) \end{pmatrix}. \tag{5}$$

(These are the inverses of the matrices given by Sakurai because he uses active rotations and we will use passive rotations.)

We will consider the three rotations in order. Under a rotation by an angle α about the x-axis, the coordinates x, y and z transform according to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_x(\alpha)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{6}$$

The matrix which transforms u' and v' to u and v is $SR_x(\alpha)S^\dagger$ which is given by:

$$SR_x(\alpha)S^\dagger = \begin{pmatrix} \cos^2(\alpha/2) & -\sin^2(\alpha/2) & -(\sqrt{2}/2)i\sin\alpha \\ -\sin^2(\alpha/2) & \cos^2(\alpha/2) & -(\sqrt{2}/2)i\sin\alpha \\ -(\sqrt{2}/2)i\sin\alpha & -(\sqrt{2}/2)i\sin\alpha & \cos\alpha \end{pmatrix} \tag{7}$$

Similarly, given the rotation matrices for the y and z axes:

$$R_y = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

and

$$R_z = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can obtain the corresponding matrices for transformation of u', v', and z':

$$SR_yS^\dagger = \begin{pmatrix} \cos^2(\beta/2) & \sin^2(\beta/2) & (\sqrt{2}/2)\sin\beta \\ \sin^2(\beta/2) & \cos^2(\beta/2) & -(\sqrt{2}/2)\sin\beta \\ -(\sqrt{2}/2)\sin\beta & (\sqrt{2}/2)\sin\beta & \cos\beta \end{pmatrix} \tag{8}$$

and

$$SR_zS^\dagger = \begin{pmatrix} \exp(i\gamma) & 0 & 0 \\ 0 & \exp(-i\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The two half derivatives which make up ψ will now be considered. In terms of u, v, and z, it is known [4] that the Laplace equation can be written as:

$$\nabla^2 f(u, v, z) = -2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial z^2} = 0. \tag{10}$$

The general solution can be written as a sum of product functions: $U(u)V(v)Z(z)$. When this expression is plugged into (10), we obtain: $U(u) = \exp(au)$, $V(v) = \exp(bv)$ and $Z(z) = \exp(\sqrt{2ab}z)$, where a and b are constants. There are also solutions of the form $g(u)$ and $h(v)$ where g and h are arbitrary functions [5]. But if g and h can be expressed as Fourier series, then even in this case f can be expressed in terms of products of exponentials.

We are now ready to evaluate the partial semiderivatives in the (u, v, z) coordinate system and compare them to the partial derivatives in a rotated coordinate system: (u', v', z'). Using the definition of Liouville [1], the half derivative of an exponential is given by $d^{1/2}C\exp ax/dx^{1/2} = C\sqrt{a}\exp ax$. In the unprimed frame,

$$(\partial^{1/2} f/\partial u^{1/2}, -\partial^{1/2} f/\partial v^{1/2}) = (\sqrt{a}, -\sqrt{b}) f. \tag{11}$$

Under a rotation about the x-axis, u, v and z are related to u', v', and z' by the matrix transformation in (7). By inverting this matrix we can write:

$$f(u, v, z) = f(\cos^2(\alpha/2)u' - \sin^2(\alpha/2)v' - (\sqrt{2}/2)i\sin\alpha z' \tag{12}$$

$$- \sin^2(\alpha/2)u' + \cos^2(\alpha/2)v' - (\sqrt{2}/2)i\sin\alpha z' \tag{13}$$

$$- (\sqrt{2}/2)i\sin\alpha u' - (\sqrt{2}/2)i\sin\alpha v' + \cos\alpha z') \tag{14}$$

$$= \exp[(\sqrt{a}\cos(\alpha/2) - i\sqrt{b}\sin(\alpha/2))^2 u' \tag{15}$$

$$+ (-i\sqrt{a}\sin(\alpha/2) + \sqrt{b}\cos(\alpha/2))^2 v' \tag{16}$$

$$+ (\sqrt{2}/2)(-ia\sin\alpha - ib\sin\alpha + 2\sqrt{ab}\cos\alpha)z']. \tag{17}$$

In this form, it is easy to obtain the two semiderivatives:

$$\partial^{1/2} f(u, v, z)/\partial u'^{1/2} = (\sqrt{a}\cos(\alpha/2) - i\sqrt{b}\sin(\alpha/2)) f(u, v, z) \tag{18}$$

and

$$\partial^{1/2} f(u, v, z)/\partial v'^{1/2} = (-i\sqrt{a}\sin(\alpha/2) + \sqrt{b}\cos(\alpha/2)) f(u, v, z). \tag{19}$$

Finally, combining these expressions with (11), we can write:

$$\begin{pmatrix} \partial^{1/2} f/\partial u'^{1/2} \\ -\partial^{1/2} f/\partial v'^{1/2} \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} \partial^{1/2} f/\partial u^{1/2} \\ -\partial^{1/2} f/\partial v^{1/2} \end{pmatrix}. \tag{20}$$

This transformation matrix is that for a spinor rotated through an angle α about the x-axis.

Similar analysis can be done for rotations about the y and z-axes: If u', v', and z' are related to u, v and z by a rotation through an angle β about the y axis then, from (8),

$$
\begin{aligned}
f(u,v,z) &= f(\cos^2(\beta/2)u' + \sin^2(\beta/2)v' + (\sqrt{2}/2)\sin\beta z' && (21)\\
&\quad + \sin^2(\beta/2)u' + \cos^2(\beta/2)v' - (\sqrt{2}/2)\sin\beta z' && (22)\\
&\quad - (\sqrt{2}/2)\sin\beta u' + (\sqrt{2}/2)\sin\beta v' + \cos\beta z') && (23)\\
&= \exp[(\sqrt{a}\cos(\beta/2) - \sqrt{b}\sin(\beta/2))^2 u' && (24)\\
&\quad + (\sqrt{a}\sin(\beta/2) + \sqrt{b}\cos(\beta/2))^2 v' && (25)\\
&\quad + (\sqrt{2}/2)(+a\sin\beta - b\sin\beta + 2\sqrt{ab}\cos\beta)z']. && (26)
\end{aligned}
$$

The semiderivatives are:

$$\partial^{1/2} f(u,v,z)/\partial u'^{1/2} = (\sqrt{a}\cos(\beta/2) - \sqrt{b}\sin(\beta/2))f(u,v,z) \tag{27}$$

and

$$\partial^{1/2} f(u,v,z)/\partial v'^{1/2} = (\sqrt{a}\sin(\beta/2) + \sqrt{b}\cos(\beta/2))f(u,v,z), \tag{28}$$

and we can write:

$$\begin{pmatrix} \partial^{1/2} f/\partial u'^{1/2} \\ -\partial^{1/2} f/\partial v'^{1/2} \end{pmatrix} = \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} \partial^{1/2} f/\partial u^{1/2} \\ -\partial^{1/2} f/\partial v^{1/2} \end{pmatrix}. \tag{29}$$

Lastly, if u', v', and z' are related to u, v and z by a rotation through an angle γ about the z axis then, from (9)

$$
\begin{aligned}
f(u,v,z) &= f(\exp(i\gamma)u', \exp(-i\gamma)v', z') && (30)\\
&= \exp[a\exp(i\gamma)u' + b\exp(-i\gamma)v' + \sqrt{2ab}z']. && (31)
\end{aligned}
$$

The two semiderivatives are:

$$\partial^{1/2} f(u,v,z)/\partial u'^{1/2} = \sqrt{a}\exp(+i\gamma/2)f(u,v,z) \tag{32}$$

and

$$\partial^{1/2} f(u,v,z)/\partial v'^{1/2} = \sqrt{b}\exp(-i\gamma/2))f(u,v,z), \tag{33}$$

so we can write:

$$\begin{pmatrix} \partial^{1/2} f/\partial u'^{1/2} \\ -\partial^{1/2} f/\partial v'^{1/2} \end{pmatrix} = \begin{pmatrix} \exp(+i\gamma/2) & 0 \\ 0 & \exp(-i\gamma/2) \end{pmatrix} \begin{pmatrix} \partial^{1/2} f/\partial u^{1/2} \\ -\partial^{1/2} f/\partial v^{1/2} \end{pmatrix}. \tag{34}$$

Thus under any rotation of the coordinate axes, the object ψ transforms as a spinor.

3 Application to the Dirac Equation

The half derivative spinors have a physical interpretation. They can be used to construct solutions to the Dirac equation for fermions with zero kinetic energy. Let functions f and g be defined by $f(u,v,z) = \exp(au + bv + 2\sqrt{ab}z)$ and $g(u,v,z) = \exp(cu + dv + 2\sqrt{cd}z)$.

As we have seen, solutions to Laplace's equation can be expressed as linear combinations of solutions of this form. Let $\Psi(u,v,z,t)$ be a four component object defined by:

$$\Psi(u,v,z,t)=\begin{bmatrix}\partial^{1/2}f/\partial u^{1/2}\\ -\partial^{1/2}f/\partial v^{1/2}\\ \partial^{1/2}g/\partial u^{1/2}\\ -\partial^{1/2}g/\partial v^{1/2}\end{bmatrix}\exp(Ct). \tag{35}$$

The Dirac equation is given by [6]:

$$\left(i\hbar\frac{\partial}{\partial t}+i\hbar c\vec{\alpha}\cdot\vec{\nabla}-\beta m_0c^2\right)\Psi=0, \tag{36}$$

with the α and β matrices given by:

$$\alpha_x=\begin{pmatrix}0 & \sigma_x\\ \sigma_x & 0\end{pmatrix},\qquad \alpha_y=\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix} \tag{37}$$

$$\alpha_z=\begin{pmatrix}0 & \sigma_z\\ \sigma_z & 0\end{pmatrix}\qquad \text{and}\qquad \beta=\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}. \tag{38}$$

Here σ_0 is the two dimensional identity matrix. In terms of derivatives with respect to u and v, the Dirac equation can be written as:

$$\left(i\hbar\frac{\partial}{\partial t}-\beta m_0c^2+i\hbar c\begin{pmatrix}0 & \sqrt{2}(\sigma_+\frac{\partial}{\partial u}-\sigma_-\frac{\partial}{\partial v})+\sigma_z\frac{\partial}{\partial z}\\ \sqrt{2}(\sigma_+\frac{\partial}{\partial u}-\sigma_-\frac{\partial}{\partial v})+\sigma_z\frac{\partial}{\partial z} & 0\end{pmatrix}\right)\Psi=0, \tag{39}$$

where σ_+ and σ_- are raising and lowering operators defined by $\sigma_+=\frac{1}{2}(\sigma_x+i\sigma_y)$ and $\sigma_-=\frac{1}{2}(\sigma_x-i\sigma_y)$. Substitution of (35) into (39) yields:

$$\begin{bmatrix}(i\hbar C-m_0c^2)\sqrt{a}f\\ -(i\hbar C-m_0c^2)\sqrt{b}f\\ (i\hbar C+m_0c^2)\sqrt{c}g\\ -(i\hbar C+m_0c^2)\sqrt{d}g\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{40}$$

There are two types of solutions then, those having $C=-im_0c^2/\hbar$ and $g\equiv 0$ (the positive energy solutions) and those having $C=+im_0c^2/\hbar$ and $f\equiv 0$ (the negative energy solutions).

4 Conclusion

Since the two half derivatives transform under rotations about the x, y, and z axes as a spinor, they constitute a spinor field analogous to the vector field generated by the gradient. These spinor fields are zero kinetic energy solutions to the Dirac equation.

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